

## DIVIDED DIFFERENCES OF INVERSE FUNCTIONS AND PARTITIONS OF A CONVEX POLYGON

MICHAEL S. FLOATER AND TOM LYCHE

ABSTRACT. We derive a formula for an  $n$ -th order divided difference of the inverse of a function. The formula has a simple and surprising structure: it is a sum over partitions of a convex polygon with  $n + 1$  vertices. The formula provides a numerically stable method of computing divided differences of  $k$ -th roots. It also provides a new way of enumerating all partitions of a convex polygon of a certain type, i.e., with a specified number of triangles, quadrilaterals, and so on, which includes Catalan numbers as a special case.

### 1. INTRODUCTION

Divided differences are a basic tool in numerical analysis: they play an important role in interpolation and approximation by polynomials and in spline theory; see [1] for a recent survey. So it is worthwhile looking for identities for divided differences that are analogous to identities for derivatives. An example is the Leibniz rule for differentiating products of functions. This rule was generalized to divided differences by Popoviciu [10, 11] and Steffensen [15]. More recently, two distinct chain rules for divided differences were derived in [5], both of which can be viewed as analogous to Faà di Bruno's formula [3, 6] for differentiating composite functions. One of the chain rules was also found in [16].

Another kind of derivative formula is a rule for differentiating the inverse of a function. Specifically, if  $y = f(x)$  and  $x = g(y)$ , then since  $y = f(g(y))$ , the chain rule for differentiation gives

$$\begin{aligned} g'(y) &= \frac{1}{f'(x)}, \\ g''(y) &= -\frac{f''(x)}{(f'(x))^3}, \\ (1) \quad g'''(y) &= -\frac{f'''(x)}{(f'(x))^4} + 3\frac{(f''(x))^2}{(f'(x))^5}, \\ g^{(4)}(y) &= -\frac{f^{(4)}(x)}{(f'(x))^5} + 10\frac{f''(x)f'''(x)}{(f'(x))^6} - 15\frac{(f''(x))^3}{(f'(x))^7}, \end{aligned}$$

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and so on. Recently, Johnson [7] derived a general formula for the  $n$ -th derivative of  $g$  (see (17)).

In this paper we derive an analogous formula for divided differences: expressing an arbitrary divided difference of  $g$  in terms of certain divided differences of  $f$ . The formula follows from the chain rule of ([5], eq. 11) and turns out to have a surprising and beautiful structure: it is a sum over partitions of a convex polygon into smaller polygons using only nonintersecting diagonals. We discuss two applications of the formula. One is that it provides a numerically stable method of computing divided differences of the square root function and other  $k$ -th root functions. Another comes from the special structure of the formula: by comparing it with Johnson's formula (17), it gives a new way of counting partitions of a convex polygon, including triangulations and the Catalan numbers as a special case.

## 2. THE INVERSE RULE

We denote by  $[x_0, x_1, \dots, x_n]f$  the usual divided difference of a real-valued function  $f$  at the real values  $x_0, \dots, x_n$ . We have  $[x_i]f = f(x_i)$ , and for distinct  $x_i$  we have  $[x_0, \dots, x_n]f = ([x_1, \dots, x_n]f - [x_0, \dots, x_{n-1}]f)/(x_n - x_0)$ . We can allow any number of the  $x_i$  to be equal to some value  $x$  if  $f$  has sufficiently many derivatives at  $x$ . In particular, if all the  $x_i$  are equal to  $x$ , then  $[x_0, x_1, \dots, x_n]f = f^{(n)}(x)/n!$ .

Suppose  $f$  is invertible in some open interval. If  $x_0$  and  $x_1$  are distinct points in this interval and  $y_i = f(x_i)$ , then  $y_0 \neq y_1$  and there is a simple formula for the first-order divided difference of  $g$ , the inverse of  $f$ , at the points  $y_0$  and  $y_1$ , namely

$$(2) \quad [y_0, y_1]g = \frac{1}{[x_0, x_1]f}.$$

If  $x_0 \neq x_1$ , this follows from the fact that

$$\frac{g(y_1) - g(y_0)}{y_1 - y_0} = \frac{x_1 - x_0}{f(x_1) - f(x_0)},$$

while if  $x_0 = x_1 = x$ , it is a restatement of the fact that  $g'(y) = 1/f'(x)$ , which holds as long as  $f$  is continuous, differentiable at  $x$  and with  $f'(x) \neq 0$ . How then might we extend this to higher-order divided differences of  $g$ ? One way is to recursively use the divided difference chain rule of ([5], eq. 11):

$$(3) \quad [y_0, \dots, y_n](f \circ g) = \sum_{k=1}^n \sum_{0=i_0 < \dots < i_k=n} [x_{i_0}, \dots, x_{i_k}]f \prod_{j=0}^{k-1} [y_{i_j}, \dots, y_{i_{j+1}}]g.$$

This formula looks complicated at first, but is in fact not so difficult to remember. We sum over the  $\binom{n-1}{k-1}$  choices of  $k-1$  strictly increasing integers  $i_1, \dots, i_{k-1}$  from the set  $\{1, 2, \dots, n-1\}$ . The product term is built up by filling the gaps between each  $y_{i_j}$  and  $y_{i_{j+1}}$ . Thus for  $n=3$  the formula looks as follows:

$$\begin{aligned} [y_0, y_1, y_2, y_3](f \circ g) &= [x_0, x_3]f [y_0, y_1, y_2, y_3]g \\ &\quad + [x_0, x_1, x_3]f [y_0, y_1]g [y_1, y_2, y_3]g \\ &\quad + [x_0, x_2, x_3]f [y_0, y_1, y_2]g [y_2, y_3]g \\ &\quad + [x_0, x_1, x_2, x_3]f [y_0, y_1]g [y_1, y_2]g [y_2, y_3]g. \end{aligned}$$

Now, equation (3) implies

$$[y_0, \dots, y_n](f \circ g) = [x_0, x_n]f [y_0, \dots, y_n]g + \sum_{k=2}^n \sum_{0=i_0 < \dots < i_k=n} [x_{i_0}, \dots, x_{i_k}]f \prod_{j=0}^{k-1} [y_{i_j}, \dots, y_{i_{j+1}}]g,$$

and since  $[y_0, \dots, y_n](f \circ g) = 0$  for  $n \geq 2$  we obtain

$$(4) \quad [y_0, \dots, y_n]g = \frac{-1}{[x_0, x_n]f} \sum_{k=2}^n \sum_{0=i_0 < \dots < i_k=n} [x_{i_0}, \dots, x_{i_k}]f \prod_{j=0}^{k-1} [y_{i_j}, \dots, y_{i_{j+1}}]g,$$

which expresses the highest-order difference of  $g$  in terms of lower-order ones. This gives in the case  $n = 2$ ,

$$[y_0, y_1, y_2]g = -\frac{[x_0, x_1, x_2]f}{[x_0, x_2]f} [y_0, y_1]g [y_1, y_2]g,$$

and applying (2) to the expressions  $[y_0, y_1]g$  and  $[y_1, y_2]g$  and using the shorthand notation

$$(5) \quad [i_1 \dots i_k] := [x_{i_1}, \dots, x_{i_k}]f,$$

we obtain

$$(6) \quad [y_0, y_1, y_2]g = -\frac{[x_0, x_1, x_2]f}{[x_0, x_1]f [x_0, x_2]f [x_1, x_2]f} = -\frac{[012]}{[01][02][12]}.$$

The case  $n = 3$  gives

$$(7) \quad [y_0, y_1, y_2, y_3]g = \frac{-1}{[03]} \left( [0123] [y_0, y_1]g [y_1, y_2]g [y_2, y_3]g + [013] [y_0, y_1]g [y_1, y_2, y_3]g + [023] [y_0, y_1, y_2]g [y_2, y_3]g \right).$$

Then using (2) and (6) to express the terms  $[y_i, y_{i+1}]g$  and  $[y_i, y_{i+1}, y_{i+2}]g$  appearing in (7) in terms of differences in  $f$ , we arrive at

$$(8) \quad [y_0, y_1, y_2, y_3]g = \frac{1}{[01][12][23][03]} \left( -[0123] + \frac{[013][123]}{[13]} + \frac{[012][023]}{[02]} \right).$$

This formula appears to have little structure at first but remarkably it can be viewed in terms of partitions of a convex polygon into smaller polygons. A partition of a convex polygon is the result of connecting any pairs of vertices with straight lines, none of which intersect. In order to see (8) in a simple way, consider the set  $\mathcal{P}$  of all possible partitions of a convex quadrilateral with ordered vertices labelled  $0, 1, 2, 3$ ; see Fig. 1. There are three partitions in  $\mathcal{P}$ , one in which no edge is inserted, and two in which one edge is inserted. We can view each partition  $\pi$  as an ordered pair  $\pi = (E, \Phi)$  with  $E = E(\pi)$  and  $\Phi = \Phi(\pi)$  denoting the sets of edges and faces in  $\pi$  respectively. Thus in the ordering of the three partitions in the figure, we have  $\mathcal{P} = \{\pi_1, \pi_2, \pi_3\}$ , where

$$(9) \quad \begin{aligned} E(\pi_1) &= \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{0, 3\}\}, & \Phi(\pi_1) &= \{\{0, 1, 2, 3\}\}, \\ E(\pi_2) &= E(\pi_1) \cup \{\{1, 3\}\}, & \Phi(\pi_2) &= \{\{0, 1, 3\}, \{1, 2, 3\}\}, \\ E(\pi_3) &= E(\pi_1) \cup \{\{0, 2\}\}, & \Phi(\pi_3) &= \{\{0, 2, 3\}, \{0, 1, 2\}\}. \end{aligned}$$

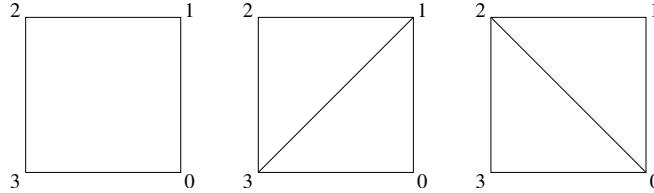


FIGURE 1. The 3 partitions of a convex quadrilateral.

With these definitions, and defining

$$[\mathbf{x}; \{i_1, i_2, \dots, i_k\}]f := [x_{i_1}, x_{i_2}, \dots, x_{i_k}]f,$$

we can write (8) simply as

$$(10) \quad [y_0, y_1, y_2, y_3]g = \sum_{i=1}^3 (-1)^{\#\Phi(\pi_i)} \frac{\prod_{\phi \in \Phi(\pi_i)} [\mathbf{x}; \phi]f}{\prod_{e \in E(\pi_i)} [\mathbf{x}; e]f},$$

where  $\#\Phi(\pi_i)$  is the cardinality of  $\Phi(\pi_i)$ , i.e., the number of faces in  $\pi$ . Each term in the sum is plus or minus a quotient whose numerator is a product of divided differences of order 2 or higher, one for each face in  $\pi_i$  and whose denominator is a product of divided differences of order 1, one for each edge in  $\pi_i$ . The quotient is multiplied by  $-1$  if the number of faces in  $\pi_i$  is odd. We will prove the general formula.

**Theorem 1.** For  $n \geq 1$ ,

$$(11) \quad [y_0, \dots, y_n]g = \sum_{\pi \in \mathcal{P}(0, \dots, n)} (-1)^{\#\Phi(\pi)} \frac{\prod_{\phi \in \Phi(\pi)} [\mathbf{x}; \phi]f}{\prod_{e \in E(\pi)} [\mathbf{x}; e]f}.$$

Here  $\mathcal{P}(0, \dots, n)$  denotes the set of partitions of a convex polygon with ordered vertices labeled  $0, 1, \dots, n$ .

Notice that the partition formula with  $n = 2$  follows from (6) because  $\mathcal{P}(0, 1, 2)$  contains the single partition,

$$\pi = (\{\{0, 1\}, \{1, 2\}, \{0, 2\}\}, \{\{0, 1, 2\}\}),$$

the only possible partition of a triangle with vertices  $0, 1, 2$ . Also, the partition formula with  $n = 1$  follows from (2) because we can view  $\mathcal{P}(0, 1)$  as containing one partition, namely

$$\pi = (\{\{0, 1\}\}, \emptyset),$$

the “trivial” partition of the line segment with endpoints  $0$  and  $1$ , which contains the edge  $e = \{0, 1\}$  itself and no faces.

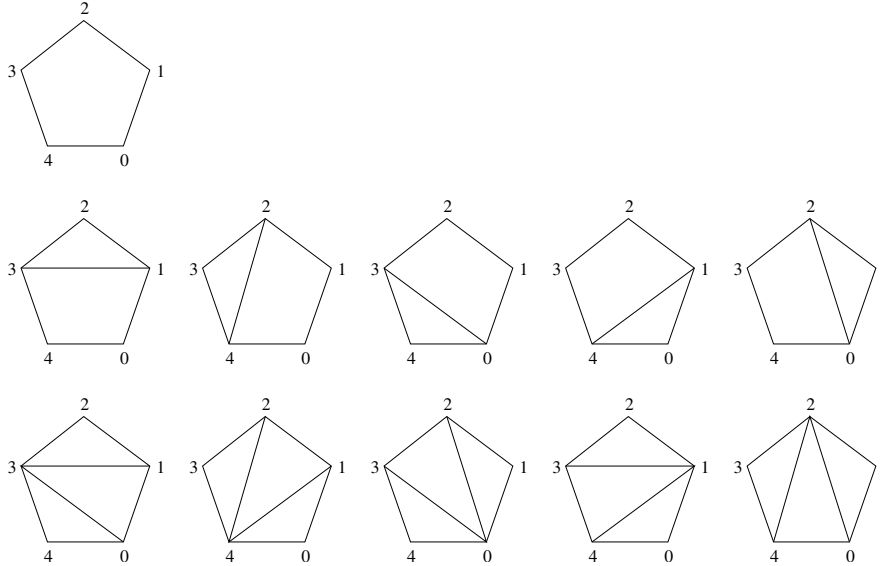


FIGURE 2. The 11 partitions of a convex pentagon.

With  $n = 4$ , the formula tells us that  $[y_0, \dots, y_4]g$  has 11 terms, one for each of the pentagonal partitions in  $\mathcal{P}(0, 1, 2, 3, 4)$ , shown in Figure 2, and we obtain

$$\begin{aligned}
 (12) \quad [y_0, \dots, y_4]g &= \left( -[01234] + \frac{[0134][123]}{[13]} + \frac{[0124][234]}{[24]} \right. \\
 &\quad + \frac{[0123][034]}{[03]} + \frac{[1234][014]}{[14]} + \frac{[0234][012]}{[02]} \\
 &\quad - \frac{[034][013][123]}{[03][13]} - \frac{[014][124][234]}{[14][24]} - \frac{[012][023][034]}{[02][03]} \\
 &\quad \left. - \frac{[014][134][123]}{[14][13]} - \frac{[012][024][234]}{[02][24]} \right) / ([01][12][23][34][04]).
 \end{aligned}$$

*Proof of Theorem 1.* The formula (11) holds for  $n = 1$ ,  $n = 2$  and  $n = 3$  by equations (2), (6) and (8). To prove it for  $n \geq 4$  we assume it holds for any smaller value of  $n$  and use induction. Starting from equation (4), note that  $k \geq 2$  in the sum, and so  $i_{j+1} - i_j \leq n - 1$  in the product and so by induction we can apply the formula (11) to all the divided differences in the product. If  $i_{j+1} - i_j = 1$  we have

$$(13) \quad [y_{i_j}, y_{i_{j+1}}]g = \frac{1}{[x_i, x_{i_{j+1}}]f},$$

while if  $i_{j+1} - i_j \geq 2$  we have

$$(14) \quad [y_{i_j}, \dots, y_{i_{j+1}}]g = \sum_{\pi \in \mathcal{P}(i_j, \dots, i_{j+1})} (-1)^{\#\Phi(\pi)} \frac{\prod_{\phi \in \Phi(\pi)} [x; \phi]f}{\prod_{e \in E(\pi)} [x; e]f}.$$

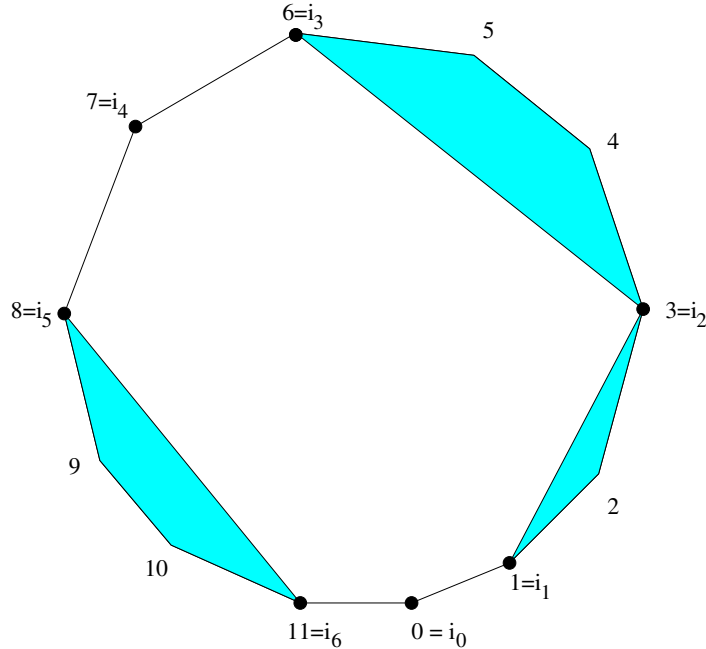


FIGURE 3. Example of a sequence  $\mathbf{i} \in I_{11}$  with three outer faces, shaded.

Let

$$I_n := \{\mathbf{i} = (i_0, i_1, \dots, i_k) : 0 = i_0 < i_1 < \dots < i_k = n, 2 \leq k \leq n\}.$$

By viewing each sequence  $\mathbf{i} \in I_n$  in the sum in (4) as a path of vertices in the convex polygon  $(0, 1, \dots, n)$ , we see that  $\mathbf{i}$  forms a special kind of partition in  $\mathcal{P}(0, 1, \dots, n)$ . Figure 3 shows a sequence in  $I_{11}$  and its partition. Each such partition contains an *inner face*, the face containing the indices  $i_0, i_1, \dots, i_k$ , and a sequence of *outer faces*, faces with vertices  $\{i_j, i_j + 1, \dots, i_{j+1}\}$  when  $i_{j+1} - i_j \geq 2$ . In Figure 3 there are three outer faces, shaded.

Now observe that each divided difference of  $g$  in (14) is a sum over all possible partitions of the outer face  $\{i_j, \dots, i_{j+1}\}$ , and since the outer faces have only vertices in common, the faces and edges of these partitions are distinct. Thus, when (13) and (14) are substituted into (4), we get

$$(15) \quad [y_0, \dots, y_n]g = \frac{-1}{[x_0, x_n]f} \sum_{\mathbf{i} \in I_n} [\mathbf{x}; \mathbf{i}]f \times \prod_{\substack{j=0 \\ i_{j+1}-i_j=1}}^{k-1} \frac{1}{[x_{i_j}, x_{i_{j+1}}]f} \sum_{\pi \in \mathcal{P}_i} (-1)^{\#\Phi(\pi)} \frac{\prod_{\phi \in \Phi(\pi)} [x; \phi]f}{\prod_{e \in E(\pi)} [x; e]f},$$

where  $\mathbf{i} = (i_0, \dots, i_k)$  and  $\mathcal{P}_i$  denotes the set of all partitions of the union of the outer faces generated by  $\mathbf{i}$ . Next observe that the first-order differences  $[x_i, x_{i_{j+1}}]f$  and  $[x_0, x_n]f$  in (15) correspond to the edges of the inner face of  $\mathbf{i}$  that are also

edges of the full polygon  $(0, 1, \dots, n)$ . Therefore

$$\begin{aligned}
 [y_0, \dots, y_n]g &= \sum_{\mathbf{i} \in I_n} \sum_{\substack{\pi \in \mathcal{P}(0,1,\dots,n) \\ \mathbf{i} \in \Phi(\pi)}} (-1)^{\#\Phi(\pi)} \frac{\prod_{\phi \in \Phi(\pi)} [\mathbf{x}; \phi] f}{\prod_{e \in E(\pi)} [\mathbf{x}; e] f} \\
 &= \sum_{\pi \in \mathcal{P}(0,1,\dots,n)} (-1)^{\#\Phi(\pi)} \frac{\prod_{\phi \in \Phi(\pi)} [\mathbf{x}; \phi] f}{\prod_{e \in E(\pi)} [\mathbf{x}; e] f}. \quad \square
 \end{aligned}$$

### 3. DIVIDED DIFFERENCES OF $k$ -TH ROOTS

It is well known that computing divided differences can be numerically unstable when some of the abscissae are close and it has been pointed out by Kahan and Fateman [8] that for certain functions the problem can be alleviated by using specific formulas. A good example is the square root function  $g(y) = \sqrt{y}$ . We might want to compute its first-order divided difference

$$[y_0, y_1]g = \frac{\sqrt{y_1} - \sqrt{y_0}}{y_1 - y_0},$$

but using this formula directly will be numerically unstable when  $y_0$  and  $y_1$  are close together. However, there is a simple solution, namely to use the alternative formula

$$[y_0, y_1]g = \frac{1}{\sqrt{y_0} + \sqrt{y_1}}.$$

Using Theorem 1 we now obtain similar formulas for divided differences of the square root function of any order. We use the fact that  $g(y) = \sqrt{y}$  is the inverse of the function  $f(x) = x^2$  and apply the formula (11). The formula simplifies because all divided differences of  $f$  of order higher than 2 are zero and

$$[x_i, x_j]f = x_i + x_j, \quad \text{and} \quad [x_i, x_j, x_k]f = 1.$$

So each term in the sum in (11) is zero whenever the partition  $\pi$  contains at least one face with four or more vertices, and the formula reduces to a sum over triangulations: partitions in  $\mathcal{P}(0, 1, \dots, n)$  in which all faces have three vertices. Every such triangulation has  $n - 1$  faces, and denoting the set of all these triangulations by  $\mathcal{T}(0, 1, \dots, n)$ , the formula becomes

$$(16) \quad [y_0, \dots, y_n]g = (-1)^{n-1} \sum_{\pi \in \mathcal{T}(0,1,\dots,n)} \frac{1}{\prod_{\{i,j\} \in E(\pi)} (x_i + x_j)},$$

where  $x_i = \sqrt{y_i}$ . The first couple of examples are

$$[y_0, y_1, y_2]g = \frac{-1}{(x_0 + x_1)(x_1 + x_2)(x_0 + x_2)},$$

$$\begin{aligned}
 [y_0, y_1, y_2, y_3]g &= \\
 &= \frac{1}{(x_0 + x_1)(x_1 + x_2)(x_2 + x_3)(x_0 + x_3)} \left( \frac{1}{(x_0 + x_2)} + \frac{1}{(x_1 + x_3)} \right).
 \end{aligned}$$

More generally, Theorem 1 provides a similar formula for divided differences of the  $k$ -th root function  $g(y) = y^{1/k}$ , for any  $k \geq 1$ , using the fact that its inverse is

$f(x) = x^k$ . It is well known that the divided difference of  $f$  is

$$[s_0, s_1, \dots, s_r]f = \sum_{0 \leq i_1 < i_2 < \dots < i_{k-r} \leq r} s_{i_1} s_{i_2} \dots s_{i_{k-r}}, \quad \text{for } 0 \leq r \leq k,$$

and  $[s_0, s_1, \dots, s_r]f = 0$  for  $r > k$ . For example, with  $g(y) = y^{1/3}$ , (11) gives

$$[y_0, y_1]g = \frac{1}{x_0^2 + x_0x_1 + x_1^2},$$

$$[y_0, y_1, y_2]g = \frac{-(x_0 + x_1 + x_2)}{(x_0^2 + x_0x_1 + x_1^2)(x_1^2 + x_1x_2 + x_2^2)(x_0^2 + x_0x_2 + x_2^2)}.$$

Using these formulas we can also derive formulas for reciprocals of  $k$ -th root functions. For a general function  $\phi$ , the reciprocal rule

$$[y_0, \dots, y_n] \frac{1}{\phi} = \sum_{k=1}^n (-1)^k \sum_{0=i_0 < \dots < i_k=n} \frac{\prod_{j=0}^{k-1} [y_{i_j}, \dots, y_{i_{j+1}}] \phi}{x_{i_0} x_{i_1} \dots x_{i_k}},$$

where  $x_i = \phi(y_i)$ , derived in [5], gives for example

$$[y_0, y_1] \frac{1}{\phi} = \frac{-1}{x_0x_1} [y_0, y_1] \phi,$$

and

$$[y_0, y_1, y_2] \frac{1}{\phi} = \frac{-1}{x_0x_2} [y_0, y_1, y_2] \phi + \frac{1}{x_0x_1x_2} [y_0, y_1] \phi [y_1, y_2] \phi.$$

So if we let  $\phi(y) = y^{1/2}$ , we obtain formulas for divided differences of the function  $g(y) = y^{-1/2}$ :

$$[y_0, y_1]g = -\frac{1}{x_0x_1} \frac{1}{x_0 + x_1},$$

$$[y_0, y_1, y_2]g = \frac{1}{x_0x_2} \frac{1}{(x_0 + x_1)(x_1 + x_2)(x_0 + x_2)} + \frac{1}{x_0x_1x_2} \frac{1}{(x_0 + x_1)} \frac{1}{(x_1 + x_2)}.$$

#### 4. DERIVATIVES OF INVERSE FUNCTIONS

It is natural to compare the formula for divided differences of inverse functions (equation (11)) with known formulas for derivatives of inverse functions. Recently Johnson [7] derived an elegant derivative formula, in terms of partitions of a set. A *partition*  $\pi$  of a set  $S$  is a collection of disjoint subsets of  $S$  whose union is  $S$ . The subsets are known as the *blocks* of the partition. Let  $Q_m$  denote the collection of all partitions of the set  $\{1, 2, \dots, m\}$  in which every block has at least size 2. Thus, for example,  $Q_2 = \{\pi\}$  where  $\pi = \{\{1, 2\}\}$ ,  $Q_3 = \{\pi\}$  where  $\pi = \{\{1, 2, 3\}\}$ , and  $Q_4 = \{\pi_1, \pi_2, \pi_3, \pi_4\}$  where

$$\pi_1 = \{\{1, 2\}, \{3, 4\}\}, \quad \pi_2 = \{\{1, 3\}, \{2, 4\}\}, \quad \pi_3 = \{\{2, 3\}, \{1, 4\}\}, \quad \text{and}$$

$$\pi_4 = \{\{1, 2, 3, 4\}\}.$$

Johnson's formula is the following.

**Theorem 2.** For  $n \geq 2$ ,

$$(17) \quad g^{(n)}(y) = \sum_{k=1}^{n-1} (-1)^k (f'(x))^{-n-k} \sum_{\substack{\pi \in Q_{n+k-1} \\ \#\pi=k}} \prod_{B \in \pi} f^{(\#B)}(x),$$

where  $\#$  denotes set cardinality.

One can check that this agrees with (1). For convenience we give a proof.

*Proof.* The formula holds for  $n = 2$  because it reduces to

$$g''(y) = -(f'(x))^{-3} f''(x).$$

Assume now that the formula holds for  $n$  and differentiate it with respect to  $x$ . This gives

$$g^{(n+1)}(y) f'(x) = A + B,$$

where

$$A = \sum_{k=1}^{n-1} (-1)^k (-n-k) (f'(x))^{-n-k-1} f''(x) \sum_{\substack{\pi \in Q_{n+k-1} \\ \#\pi=k}} \prod_{B \in \pi} f^{(\#B)}(x),$$

and

$$B = \sum_{k=1}^{n-1} (-1)^k (f'(x))^{-n-k} \sum_{\substack{\pi \in Q_{n+k-1} \\ \#\pi=k}} \sum_{C \in \pi} f^{(\#C+1)} \prod_{\substack{B \in \pi \\ B \neq C}} f^{(\#B)}(x).$$

Now if  $B(j; \pi)$  denotes the block of  $\pi$  that contains the element  $j$ , we have

$$\begin{aligned} A &= \sum_{k=2}^n (-1)^k (n+k-1) (f'(x))^{-n-k} f''(x) \sum_{\substack{\pi \in Q_{n+k-2} \\ \#\pi=k-1}} \prod_{B \in \pi} f^{(\#B)}(x) \\ &= \sum_{k=2}^n (-1)^k (f'(x))^{-n-k} \sum_{\substack{\pi \in Q_{n+k} \\ \#\pi=k \\ \#B(n+k;\pi)=2}} \prod_{B \in \pi} f^{(\#B)}(x), \end{aligned}$$

because there are  $n+k-1$  ways of forming a block of size 2 from the set  $\{1, 2, \dots, n+k\}$  when  $n+k$  is included, and

$$B = \sum_{k=1}^{n-1} (-1)^k (f'(x))^{-n-k} \sum_{\substack{\pi \in Q_{n+k} \\ \#\pi=k \\ \#B(n+k;\pi)>2}} \prod_{B \in \pi} f^{(\#B)}(x).$$

Then summing  $A$  and  $B$  and dividing by  $f'(x)$  gives the right-hand side of (17) with  $n$  replaced by  $n+1$ . □

Next we transform this derivative formula into an alternative formula which, in the spirit of Faà di Bruno's formula for composite functions, is over integer partitions and is free of repeated terms.

**Corollary 1.** For  $n \geq 2$ ,

$$(18) \quad g^{(n)}(y) = \sum_{k=1}^{n-1} \sum_{\mathbf{b}} \frac{(-1)^k}{(f'(x))^{n+k}} \frac{(n+k-1)!}{b_2! \cdots b_n!} \left( \frac{f''(x)}{2!} \right)^{b_2} \cdots \left( \frac{f^{(n)}(x)}{n!} \right)^{b_n},$$

where the second sum is over nonnegative solutions  $\mathbf{b} = (b_2, \dots, b_n)$  to

$$(19) \quad b_2 + b_3 + \cdots + b_n = k \quad \text{and} \quad 2b_2 + 3b_3 + \cdots + nb_n = n+k-1.$$

Subtracting the sum on the left in (19) from the sum on the right we see that the double sum in (18) is over all nonnegative integer solutions  $b_2, \dots, b_n$  to the equation

$$(20) \quad b_2 + 2b_3 + \dots + (n - 1)b_n = n - 1.$$

The only solution of (20) with  $b_n \neq 0$  is  $b_n = 1$ , and  $b_2 = \dots = b_{n-1} = 0$  leading to  $k = 1$  and the term  $-f^{(n)}(x)/(f'(x))^{n+1}$ . All other solutions have  $b_n = 0$  and we only have  $n - 2$  unknowns in (20). As an example, for  $n = 4$  the reduced equation is  $b_2 + 2b_3 = 3$  and we easily find two solutions. One is  $b_2 = b_3 = 1, b_4 = 0$  giving  $k = 2$  and the term  $10f''(x)f'''(x)/(f'(x))^6$ . The other solution is  $b_2 = 3, b_3 = b_4 = 0$  corresponding to  $k = 3$  and we get the term  $-15(f''(x))^3/(f'(x))^7$ . Thus we obtain the formula for the 4th derivative in (1).

*Proof.* As observed by Johnson [7], starting from (17), instead of summing over partitions of sets, we can instead sum over the (ordered) block sizes of these partitions, and we arrive at

$$(21) \quad g^{(n)}(y) = \sum_{k=1}^{n-1} \frac{(-1)^k}{(f'(x))^{n+k}} \frac{(n+k-1)!}{k!} \times \sum_{\substack{j_1+\dots+j_k=n+k-1 \\ j_i \geq 2}} \left( \frac{f^{(j_1)}(x)}{j_1!} \right) \dots \left( \frac{f^{(j_k)}(x)}{j_k!} \right),$$

because  $(n+k-1)!/(j_1! \dots j_k!)$  is the number of set partitions with blocks of sizes  $j_1, \dots, j_k$ , and  $k!$  is the number of permutations of the sequence  $j_1, \dots, j_k$ . Now (21) contains repeated terms. Two terms in the sum in (21) are the same whenever their corresponding sequences  $(j_1, \dots, j_k)$  contain the same number of twos, the same number of threes, and so on. The largest possible number appearing in the sequence is  $n$ : in the case  $k = 1$ , we have  $j_1 = n$ . Thus we let  $b_2$  be the number of twos,  $b_3$  the number of threes, and so on up to  $b_n$ , the number of  $n$ 's, and we obtain (19). Then, since the number of positive integer solutions to the equation  $j_1 + \dots + j_k = n + k - 1$  containing  $b_2$  twos,  $b_3$  threes, and so on up to  $b_n$   $n$ 's is the multinomial coefficient  $k!/(b_2! \dots b_n!)$ , it follows that  $g^{(n)}(y)$  can be expressed as (18) where the sum is over all solutions  $\mathbf{b}$  to (19).  $\square$

### 5. COUNTING PARTITIONS OF CONVEX POLYGONS

Another application of the divided difference inverse formula of Theorem 1 is to counting partitions of a convex polygon of a specified type. Such enumeration problems go back to Euler, Catalan, and Cayley [4]; see Przytycki and Sikora [12]. Consider first enumerating triangulations.

**Theorem 3.** *The number of triangulations of a convex  $(n + 1)$ -gon is the Catalan number*

$$(22) \quad C_{n-1} = \frac{1}{n} \binom{2n-2}{n-1}.$$

Various known ways of proving this formula are covered in Chapter 7 of Stanley's book [14].

*Proof.* We prove the formula by specializing the inverse formula (11) to the square root function  $g(y) = \sqrt{y}$ . As in Section 3, this gives equation (16), which is a sum over triangulations. Let the points  $x_0, x_1, \dots, x_n$  in the latter formula converge to some positive value  $x > 0$ . Since any triangulation in  $\mathcal{T}(0, 1, \dots, n)$  contains  $2n - 1$  edges, we then obtain

$$\frac{1}{n!} \frac{d^n}{dy^n} y^{1/2} = (-1)^{n-1} \sum_{\pi \in \mathcal{T}(0,1,\dots,n)} \frac{1}{(2x)^{2n-1}} = \frac{(-1)^{n-1}}{(2x)^{2n-1}} C_{n-1},$$

and so

$$\frac{d^n}{dy^n} y^{1/2} = \frac{(-1)^{n-1}}{2^{2n-1}} n! C_{n-1} \frac{1}{y^{n-1/2}}.$$

On the other hand,  $n$ -fold differentiation of  $\sqrt{y}$  with respect to  $y$  gives

$$\frac{d^n}{dy^n} y^{1/2} = \prod_{p=0}^{n-1} (1/2 - p) \frac{1}{y^{n-1/2}} = \frac{(-1)^{n-1} (2n - 2)!}{2^{2n-1} (n - 1)!} \frac{1}{y^{n-1/2}}.$$

Comparing these two formulas gives (22). □

Next consider more general partitions of a convex polygon. We will say that a partition of a convex  $(n + 1)$ -gon is of type  $\mathbf{b} = (b_2, b_3, \dots, b_n)$  if it contains  $b_2$  triangles,  $b_3$  quadrilaterals, and so on, and in general  $b_i$   $(i + 1)$ -gons. Note that the  $b_i$  necessarily satisfy the formulas (19), where  $k$  is the total number of faces. This can easily be proved by induction on  $k$ . If  $k = 1$ , there is just one polygon, an  $(n + 1)$ -gon, and so  $b_2 = \dots = b_{n-1} = 0$  and  $b_n = 1$ , and the equation holds. Otherwise if  $k \geq 2$ , there is at least one diagonal, and by removing the diagonal, and applying the formulas to the reduced partition, one easily deduces (19).

We will use the divided difference inverse formula, combined with the derivative formula of Corollary 1 to deduce

**Theorem 4.** *The number of partitions of a convex  $(n + 1)$ -gon of type  $\mathbf{b}$ , satisfying (19), is*

$$(23) \quad C(\mathbf{b}) = \frac{(n + k - 1)(n + k - 2) \cdots (n + 1)}{b_2! b_3! \cdots b_n!}.$$

This includes the Catalan number (22) as the special case in which  $b_2 = n - 1$  and  $b_i = 0$  for  $i = 3, \dots, n$ .

*Proof.* Letting  $x_0, \dots, x_n$  in (11) converge to some  $x$  in a neighbourhood in which  $g$  has a continuous  $n$ -th derivative implies

$$(24) \quad \frac{g^{(n)}(y)}{n!} = \sum_{\pi \in \mathcal{P}(0,1,\dots,n)} \frac{(-1)^{\#\Phi(\pi)}}{(f'(x))^{\#E(\pi)}} \prod_{\phi \in \Phi(\pi)} \frac{f^{(\#\phi-1)}(x)}{(\#\phi-1)!}.$$

Now the product in (24) is the same for all partitions  $\pi$  that contain the same number of triangles, the same number of quadrilaterals, and so on. Thus to avoid repeated terms we sum instead over  $b_2$ , the number of triangles,  $b_3$ , the number of quadrilaterals, and so on, where the numbers  $b_2, \dots, b_n$  satisfy (19). Then noting that  $\#E(\pi) = n + \#\Phi(\pi)$ , we can rewrite (24) as

$$(25) \quad \frac{g^{(n)}(y)}{n!} = \sum_{k=1}^{n-1} \sum_{\mathbf{b}} \frac{(-1)^k}{(f'(x))^{n+k}} C_{n-1}(\mathbf{b}) \left( \frac{f''(x)}{2!} \right)^{b_2} \cdots \left( \frac{f^{(n)}(x)}{n!} \right)^{b_n},$$

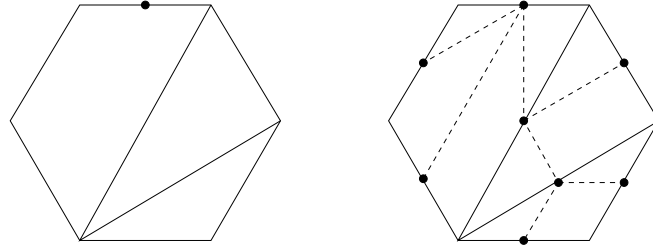


FIGURE 4. Bijection between polygon partitions and rooted planar trees.

where the second sum is over all solutions  $\mathbf{b}$  to (19).

Now choose any real values  $c_2, c_3, \dots, c_n$ , and let  $f$  be the polynomial

$$f(x) = x + \sum_{i=2}^n c_i x^i.$$

Since  $f'(0) = 1$ ,  $f$  is invertible in a small enough neighbourhood of  $x = 0$ , with inverse  $g$ , and so both formulas (25) and (18) are valid for this  $f$ . So, subtracting  $(1/n!)$  times (18) from (25) and setting  $x = 0$  gives

$$(26) \quad \sum_{k=1}^{n-1} \sum_{\mathbf{b}} \left( C_{n-1}(\mathbf{b}) - \frac{(n+k-1)!}{b_2! \cdots b_n! n!} \right) c_2^{b_2} \cdots c_n^{b_n} = 0,$$

where the sum is over nonnegative solutions to (19). But the left-hand side is a linear combination of multinomials in the variables  $c_2, \dots, c_n$ , and since they are linearly independent their coefficients must all be zero, and this proves (23).  $\square$

There are other ways of proving the formula (23). One way is to transform the problem into that of enumerating planar trees, and use a known solution to that. First note that there is a one-to-one correspondence between partitions of a convex polygon and rooted planar trees, as explained on page 171 of Stanley [14], and illustrated in Fig 4. One edge of the  $(n+1)$ -gon is fixed and forms the root of the tree which is grown recursively through the faces of the polygon partition. We say that the tree has type  $\mathbf{r} = (r_0, r_1, r_2, \dots)$  if  $r_i$  is the number of nodes with  $i$  successors. Stanley shows on page 34 of [14] that the number of rooted planar trees of type  $\mathbf{r}$  is

$$(27) \quad P(\mathbf{r}) = \frac{1}{N} \binom{N}{r_0, r_1, \dots, r_{N-1}},$$

where  $N = \sum_i r_i$  (the total number of nodes in the tree) and  $\sum_i (1-i)r_i = 1$ . Suppose now that we generate a tree from a convex  $(n+1)$ -gon of type  $\mathbf{b}$  with  $k = \sum_i b_i$ . Then  $N = n+k$ ,  $r_0 = n$ ,  $r_1 = 0$ , and  $r_i = b_i$  for  $i = 2, 3, \dots$ , and substituting these values into (27) gives (23). Another way to deduce formula (23) is to use the theory of noncrossing partitions. A partition of the set  $\{1, 2, \dots, p\}$  is said to be noncrossing if whenever four elements,  $1 \leq a < b < c < d \leq p$ , are such that  $a$  and  $c$  belong to the same block and  $b$  and  $d$  belong to the same block, the two blocks coincide. For example, on the right of Figure 5 we see a graphical illustration of the noncrossing partition

$$\{\{1, 2, 3\}, \{4, 7\}, \{5, 6\}\}.$$

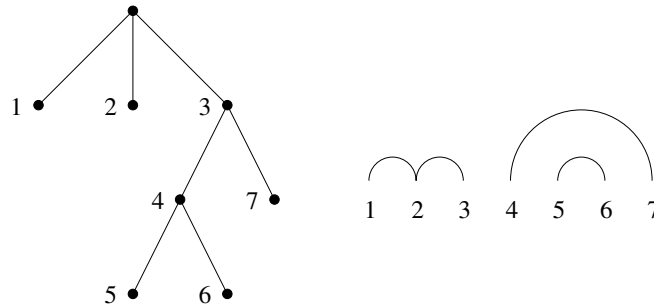


FIGURE 5. Bijection between planar trees and noncrossing partitions.

Let us say that a noncrossing partition is of type  $\mathbf{m} = (m_1, m_2, \dots, m_p)$  if it contains  $m_i$  blocks of size  $i$ . Using an inductive proof, Kreweras [9] (see also Simion [13]) showed that the number of noncrossing partitions of type  $\mathbf{m}$  is

$$(28) \quad NC(\mathbf{m}) = \frac{p(p-1) \cdots (p-b+1)}{m_1! m_2! \cdots m_p!},$$

where  $b = m_1 + \cdots + m_p$ .

Now there is a correspondence between planar trees and noncrossing partitions, found by Dershowitz and Zaks [2]. We label all the nonroot nodes of the tree recursively, starting from the root and always labelling the next left-most subtree. If  $N$  is the number of nodes in the tree, there are  $p = N - 1$  labelled nodes, and we form a block of  $\{1, 2, \dots, p\}$  by gathering together all nodes of the tree that share a common parent. The bijection is illustrated in Fig 5. Using the rooted tree generated by a polygon partition of an  $(n + 1)$ -gon of type  $\mathbf{b}$ , with  $k = \sum_i b_i$ , we have  $p = N - 1 = n + k - 1$ ,  $m_1 = 0$ ,  $m_i = b_i$ ,  $i = 2, 3, \dots$ , and  $b = k$ , and substituting these values into (28) again yields (23).

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CENTRE OF MATHEMATICS FOR APPLICATIONS, DEPARTMENT OF INFORMATICS, UNIVERSITY OF OSLO, P.O. BOX 1053, BLINDERN, 0316 OSLO, NORWAY

CENTRE OF MATHEMATICS FOR APPLICATIONS, DEPARTMENT OF INFORMATICS, UNIVERSITY OF OSLO, P.O. BOX 1053, BLINDERN, 0316 OSLO, NORWAY